Journal of Engineering Mathematics, Vol. 3, No. 2, April 1969
Wolters-Noordhoff Publishing – Groningen
Printed in the Netherlands

Some Applications for Kramer's Generalized Sampling Theorem

A. J. JERRI

Clarkson College of Technology, Potsdam, New York 13676, U.S.A.

(Received October 8, 1968)

SUMMARY
Kramer's generalization of Shannon's sampling theorem takes us from a signal represented by a finite Fourier transform to a signal represented by another and more general finite integral transform. In this paper we will attempt to show that the already obtained results for Kramer's theorem are of use in the field of finite integral transforms. Also by introducing such transforms one can treat some communications problems. An example is the case of representing a signal which is the output of time variant filter.

1. Introduction

Kramer's [1] generalized sampling theorem statement is "Let I be an interval. Suppose that for each real t

$$f(t) = \int_I K(t, x) g(x) dx \,, \tag{1}$$

where $g(x) \in L_2(I)$. Suppose that for each real t, $K(t, x) \in L_2(I)$ and that there exists a countable set $E = \{t_n\}$ such that $\{K(t_n, x)\}$ is a complete orthogonal set on $L_2(I)$. Then

$$f(t) = \lim_{N \to \infty} \sum_{|n| \leqq N} f(t_n) S_n(t) \,, \tag{2}$$

where

$$S_n(t) = \frac{\int_I K(t, x) \overline{K(t_n, x)} dx}{\int_I |K(t_n, x)|^2 dx} \,. \tag{3}$$"

Now the Shannon's [2] sampling theorem becomes its special case of $K(t, x) = e^{itx}$. Kramer showed that these conditions on the kernel $K(t, x)$ are exhibited by the solution of nth order, self-adjoint differential equation and gave a first order and the Bessel differential equation as illustrations.

Campbell [3] considered as kernels of the generalized sampling theorem the solution of a regular first order and regular second order differential equation with separated boundary conditions, and the solutions of the singular Bessel and Legendre equations. In [4] we considered the associated Legendre, the Gegenbauer, the Tchebichef, the prolate spheroidal functions and a Bessel-like function which is a solution to a fourth order differential equation. Although this theorem represents an important extension to Shannon's sampling theorem, we notice that it was neglected, especially in applications. This is no surprise when we know that the communication engineer uses the Fourier transform mainly, and that the ideal low-pass filter, with its nice time invariant impulse response, is used to give the physical interpretation. This paper is devoted to the possible reason for this neglect and at the same time will offer some suggestions in this direction. We will attempt to raise some questions, to show that the Kramer generalization of the sampling theorem might be a very handy tool. First, this generalization takes us from a signal represented by a finite Fourier transform to a signal represented by another and more general finite integral transform. Hence, the results that have already been obtained in

[1], [3] and [4] for Kramer's theorem may prove to be of use in the field of finite integral transforms. In addition, introducing such transforms might simplify some communication problems as it did in other fields. This last question may deal with the possibility, for example, of using the generalized sampling theorem for the case of a signal which is the output of a time variant filter. We now note that if such integral transforms are introduced then Kramer's theorem will play a role similar to the one played by Shannon's theorem in terms of the finite Fourier transform.

2. Other Finite Integral Transforms for Communication Theory

Here, we first introduce two finite integral transforms as they appear in the literature [5, 6, 7], namely the finite Hankel and Legendre transforms, then try to extend their definition with the help of Kramer's theorem.

Finite Hankel Transform

$$\bar{f}(t_p) = \int_0^1 f(r) r J_n(t_p r) dr , \qquad J_n(t_p) = 0, \; p = 1, 2, \dots , \tag{4}$$

with the inverse transform

$$f(r) = \sum_{p=1}^{\infty} \frac{2\bar{f}(t_p)}{J_{n+1}^2(t_p)} J_n(t_p r) . \tag{5}$$

Let us extend $\bar{f}(t_p)$ to $\bar{f}(t)$ in (4) with unrestricted t. We quickly realize that $\bar{f}(t)$ can be automatically calculated in terms of $\bar{f}(t_p)$ by the use of Kramer's theorem, which gives

$$\bar{f}(t) = \sum_{p=1}^{\infty} \bar{f}(t_p) S_p(t) , \tag{6}$$

where $S_p(t)$ is given in [1].

Legendre Transform

$$\bar{f}(n) = \int_{-1}^{1} f(u) P_n(u) du \tag{7}$$

and

$$f(u) = \sum_{n=0}^{\infty} (n+\tfrac{1}{2}) \bar{f}(n) P_n(u) . \tag{8}$$

We extend $\bar{f}(n)$ to $\bar{f}(v)$ and in the same way as above we get $\bar{f}(v)$ in terms of $\bar{f}(n)$, with the aid of Kramer's theorem to be

$$\bar{f}(v) = \sum_{n=-\infty}^{\infty} \bar{f}(n) S_n(v) , \tag{9}$$

where $S_n(v)$ is given in [3]. In the same way, this extension can be applied to all finite integral transforms with kernel defined by (1) and where $S_n(t)$ is found in [3] and [4].

Next we take the above two examples of finite integral transforms to show their possible advantage for the system function analysis of filters. In (4) let the inverse Hankel transform of $\bar{f}(t)$ with $n=2$ be $f(r)=r$. In contrast to this the inverse Fourier transform to the same $\bar{f}(t)$ is found in [4] to be

$$f(r) = \frac{(1-r^2)^{\frac{1}{2}}(2-3r-r^2)}{15\pi} . \tag{10}$$

This, obviously, is a more complicated system function than $f(r)=r$.

Now in (7) let $f(u)=1$, $u=\cos\theta$, be the inverse Legendre transform of $\bar{f}(v)$, then we can show [4] that the corresponding inverse Fourier transform of $\bar{f}(v)$ is

$$f(u) = 2^{\frac{3}{2}} \cos \frac{u}{2} \tag{11}$$

again a more complicated system function than $f(u) = 1$.

The third question deals with a problem of a different nature, concerning the fact that most linear filters are treated as stationary ones [8]. That is, their impulse response depends on $(t-t')dt'$. For example, the output

$$x(t) = \int_{-\infty}^{\infty} k(t, t') f(t') dt' \tag{12}$$

with

$$k(t, t') \equiv k(t-t'). \tag{13}$$

A more specific example is the impulse response of an ideal low-pass filter

$$k(t-t') = \frac{\sin a(t-t')}{\pi(t-t')}. \tag{14}$$

Here we notice the spirit of the Fourier Transform and its corresponding convolution theorem. The question here is that Kramer's generalization considers $K(t, t')$ in general and not only $k(t-t')$. As such it might very well be found to be of interest in the field of time variant linear filter analysis or its output signal sampling.

Another example that might show an advantage of Kramer's sampling theorem is given that

$$f(t_1, t_2) = \frac{1}{2\pi} \int_{-a_2}^{a_2} \int_{-a_1}^{a_1} g(x_1, x_2) e^{i(x_1 t_1 + x_2 t_2)} dx_1 dx_2, \tag{15}$$

where

$$g(x_1, x_2) = 0, \qquad |x_1| > |a_1|, \ |x_2| > |a_2|, \tag{16}$$

is a two-dimensional finite Fourier transform. For this we need a product of two infinite series [9] to represent $f(t_1, t_2)$ in terms of its sample points, $f(n_1 \pi/a_1, n_2 \pi/a_2)$. But if we have $f(t)$ with $\rho = (t_1^2 + t_2^2)^{\frac{1}{2}}$ and such that $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$ then from [6] we can write (15) as

$$\bar{f}(\rho) = \int_0^b r f(r) J_0(\rho r) dr, \tag{17}$$

a finite Hankel transform. Then, using Kramer's theorem, $\bar{f}(\rho)$ can be represented by an infinite series in terms of its sample points $\bar{f}(\rho_n)$.

REFERENCES

[1] H. P. Kramer; A generalized sampling theorem, *Jour. Math. Phys.*, 38, pp. 68–72, 1959.

[2] C. E. Shannon; Communications in the presence of noise, *Proc. I.R.E.*, 37, pp. 10–21, 1949.

[3] L. L. Campbell; A comparison of the sampling theorems of Kramer and Whittaker, *Jour. Soc. Indust. App. Math.*, 12, pp. 117–130, 1964.

[4] A. J. Jerri; On extension of the generalized sampling theorem, Ph.D. Thesis (*Tech. Report* No. 33), Dept. of Mathematics, Oregon State University, Corvallis, Oregon, 1967.

[5] I. N. Sneddon; Finite Hankel transforms, *Philosophical Magazine*, 37, pp. 17–25, 1946.

[6] I. N. Sneddon; *Fourier transforms*, McGraw-Hill, New York, 1951.

[7] C. J. Tranter; *Integral transforms in mathematical physics*, Wiley, New York, 1951.

[8] L. A. Wainstein and V. D. Zubakov; *Extraction of signals from noise*, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.

[9] E. Parzen; A simple proof and some extensions of sampling theorems, *Technical report* No. 7, Department of Statistics, Stanford University, Stanford, California, 1956.